# GROSS-HOPKINS DUALITY AND THE GORENSTEIN CONDITION

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Abstract. Gross and Hopkins have proved that in chromatic stable homotopy, Spanier-Whitehead duality nearly coincides with Brown-Comenetz duality. Our goal is to give a conceptual interpretation for this phenomenon in terms of the Gorenstein condition for maps of ring spectra in the sense of [5] We describe a general notion of Brown-Comenetz dualizing module for a map of ring spectra and show that in this context such dualizing modules correspond bijectively to invertible $K(n)$-local spectra.

## 1. Introduction

Suppose that $\mathbb{S}$ is the sphere spectrum and $\mathbb{I}$ its Brown-Comenetz dual. The Spanier-Whitehead dual $D_{\mathbb{S}}X$ of a spectrum $X$ is defined to be the mapping spectrum $\mathrm{Map}(X, \mathbb{S})$, while the Brown-Comenetz dual $D_{\mathbb{I}}X$ is the spectrum $\mathrm{Map}(X, \mathbb{I})$. These are very different from one another: for instance, Spanier-Whitehead duality behaves well on homology (if $X$ is finite then $H_i(D_{\mathbb{S}}X) \cong H^{-i}X$), while Brown-Comenetz duality behaves well on homotopy ($\pi_i(D_{\mathbb{I}}X) \cong \mathrm{Hom}(\pi_{-i}X, \mathbb{Q}/\mathbb{Z})$).

Nevertheless, Gross and Hopkins [10] have proved that in some localized stable homotopy situations, the appropriate version of Spanier-Whitehead duality nearly coincides with Brown-Comenetz duality. Our goal is to give a conceptual interpretation for this phenomenon. To do this we describe the notion of a *Brown-Comenetz dualizing module* $\mathcal{I}$ (1.5) for a ring spectrum map $R \to k$. There are two common mechanisms for the construction of a dualizing module:

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(1) if $R \to k$ is Gorenstein (1.10), there is a way to obtain a dualizing module $\mathcal{G}$ from $R$ itself. The duality functor $D_{\mathcal{G}}$ over $R$ agrees with Spanier-Whitehead duality over $R$ (1.11);
(2) if $R$ satisfies a different (milder) condition, there is a"trivial" dualizing module $\mathcal{I}_0$ constructed by coinduction (1.8) from $\mathbb{I}$ and the unit map $\mathbb{S} \to R$. The duality functor $D_{\mathcal{I}_0}$ over $R$ then agrees with garden-variety Brown-Comenetz duality over $\mathbb{S}$.

When both of the above constructions go through, the question of whether $\mathcal{G} \sim \mathcal{I}_0$, or equivalently of whether $D_{\mathcal{G}} \sim D_{\mathcal{I}_0}$, is a type of orientability issue (1.13). In the basic case of the sphere spectrum, only (2) applies, and so there is no call to measure Brown-Comenetz duality against Spanier-Whitehead duality by comparing the kind of Brown-Comenetz dualizing modules we consider. Things are different in the context of Gross–Hopkins duality. In this case *both* $\mathcal{G}$ and $\mathcal{I}_0$ exist, and although they do not quite agree, they are very similar to one another. We indicate in 1.20 why $\mathcal{G}$ can be distinguished from $\mathcal{I}_0$ by the algebraic calculation made in [9], and interpret this as closely analogous to distinguishing two spherical fibrations by calculating their Stiefel-Whitney classes. We classify all of the Brown–Comenetz dualizing modules in this case (1.25), and explain why they correspond bijectively to invertible modules.

In describing our point of view, we start with the general notion of Brown-Comenetz duality and use this to describe the homotopical form of Gorenstein duality [5]. To put these ideas in a more familiar homotopy theoretic context, we point out that Poincaré duality is a special case of Gorenstein duality. Finally we indicate how Gross-Hopkins duality fits into this framework. This paper could not have been written without [13] and [17]; a lot of what we do is to give a different slant to the material in [17]. Although our treatment has an intrinsic interest, it can also be viewed as an extended example of the theory of [5], an example which highlights the importance of orientability issues.

1.1. *Some notation.* We refer to a ring spectrum $R$ as an $\mathbb{S}$-*algebra*, and a module spectrum over $R$ as an $R$-*module* [6] [12]. A map between spectra is an *weak equivalence* (*equivalence* for short) if it induces an isomorphism on homotopy groups. If $M$, $N$ are left $R$-modules, then $\mathrm{Hom}_R(M, N)$ denotes the spectrum of (derived) $R$-module maps between them; if $M$ is a left $R$-module and $N$ a right $R$-module, then $N \otimes_R M$ is the (derived) smash product of $M$ and $N$ over $R$.

There's no harm in treating an ordinary ring $R$ as an $\mathbb{S}$-algebra. In that case a *module* over $R$ in our sense corresponds to what is usually

called a chain complex over $R$, $\mathrm{Hom}_R(M, N)$ to the derived mapping complex, and $N \otimes_R M$ to the derived tensor product. If $R$ is an ordinary ring and $M$, $N$ are ordinary left $R$-modules, treated as chain complexes concentrated in degree 0, then according to our conventions $\mathrm{Hom}_R(M, N)$ is a spectrum with $\pi_i \mathrm{Hom}_R(M, N) \cong \mathrm{Ext}_R^{-i}(M, N)$. Similarly, if $N$ is an ordinary right $R$-module, then $N \otimes_R M$ is a spectrum with $\pi_i(N \otimes_R M) \cong \mathrm{Tor}_i^R(N, M)$. In these cases we write $\mathrm{Ext}_R^0(M, N)$ for the usual group of homomorphisms $M \to N$, and $N \oslash_R M = \mathrm{Tor}_0^R(N, M)$ for the usual tensor product.

If $R$ is an ordinary ring with a distinguished maximal ideal $\mathfrak{m}$, we will refer to an ordinary finitely generated $\mathfrak{m}$-primary torsion $R$-module as a *finite length $R$-module.*

If $R$ is an $\mathbb{S}$-algebra and $k$, $M$ are $R$-modules, then $\mathrm{Cell}_k(M)$ denotes the *$k$-cellular approximation* of $M$: $\mathrm{Cell}_k(M)$ is built from $k$ (2.2), and there is a map $\mathrm{Cell}_k(M) \to M$ which is a $\mathrm{Cell}_k$*-equivalence*, i.e., induces an equivalence on $\mathrm{Hom}_R(k, –)$.

1.2. **Brown-Comenetz duality.** Suppose that $R \to k$ is a map of $\mathbb{S}$-algebras. Let $\mathcal{E}$ be the derived endomorphism $\mathbb{S}$-algebra $\mathrm{End}_R(k)$. An $R$-module $M$ is said to be *effectively constructible from $k$* if the natural evaluation map

$$\mathrm{Hom}_R(k, M) \otimes_{\mathcal{E}} k \to M \tag{1.3}$$

is an equivalence (cf. 2.5).

1.4. *Remark.* If $M$ is effectively constructible from $k$ then $M$ is built from $k$ as an $R$-module. For some $R$ and $k$, the converse holds (2.7).

1.5. **Definition.** A *Brown-Comenetz dualizing module* for $R \to k$ is an $R$-module $\mathcal{I}$ which is effectively constructible from $k$ and has the property that, for some $d \geq 0$, $\mathrm{Hom}_R(k, \mathcal{I})$ is equivalent as a left $k$-module to $\Sigma^d k$.

Giving such a dualizing module $\mathcal{I}$ involves finding a way of extending to $R$-modules the notion of ordinary (i.e., Spanier-Whitehead) duality for $k$-modules. As 1.3 suggests, in favorable cases [5, 6.9] these dualizing modules correspond to appropriate right $\mathcal{E}$-module structures on (a suspension of) $k$.

1.6. *Examples (uniqueness).* [5, §5] The module $\mathbb{Z}/p^\infty$ is a Brown-Comenetz dualizing module for $\mathbb{Z} \to \mathbb{Z}/p$. The $p$-primary summand of the spectrum $\mathbb{I}$ is a Brown-Comenetz dualizing module for $\mathbb{S} \to \mathbb{Z}/p$. In both of these cases, up to suspension and equivalence there is only one Brown-Comenetz dualizing module for $R \to k$.

1.7. *Examples (non–uniqueness).* Let $X$ be a 1-connected based finite CW-complex. Let $k$ denote $\mathbb{S}$, and let $R = C^*(X;k)$ denote the Spanier-Whitehead dual (over $\mathbb{S}$) of the unreduced suspension spectrum of $X$. Then $R$ is an $\mathbb{S}$-algebra under a multiplication induced by the diagonal map, and there is an augmentation $R \to k$ given by restriction to be basepoint of $X$. As in 1.13, Brown–Comenetz dualizing modules for $R \to k$ correspond bijectively up to equivalence to stable spherical fibrations over $X$.

1.8. *Examples (Coinduction).* Suppose that $T \to R$ is a map of $\mathbb{S}$-algebras, and that $J$ is a Brown-Comenetz dualizing module for $T \to k$. Let $\mathcal{I} = \mathrm{Cell}_k^R \mathrm{Hom}_T(R, J)$. If $\mathcal{I}$ is effectively constructible from $k$, then $\mathcal{I}$ is a Brown-Comenetz dualizing module for $R \to k$, called the Brown-Comenetz dualizing module *coinduced* from $J$.

1.9. **Gorenstein duality.** Let $f : R \to k$ be as above.

1.10. **Definition.** [5, 8.1] The map $f : R \to k$ is *Gorenstein* if $\mathrm{Cell}_k(R)$ is a Brown-Comenetz dualizing module for $f$.

1.11. *Remark.* Suppose $R \to k$ is Gorenstein, with associated Brown-Comenetz dualizing module $\mathcal{G} = \mathrm{Cell}_k(R)$. The map $\mathcal{G} \to R$ induces an equivalence $\mathrm{Hom}_R(M, \mathcal{G}) \to \mathrm{Hom}_R(M, R)$ for $M = k$ and thus for any $R$-module $M$ built from $k$. For such $M$, this gives an equivalence

$$D_{\mathcal{G}}M \sim D_R M\,.$$

In other words, if $R \to k$ is Gorenstein, then for $R$-modules which are built from $k$, Spanier-Whitehead duality agrees with the variant of Brown-Comenetz duality singled out by the Gorenstein condition.

1.12. *Example (algebra).* Suppose that $R$ is the formal power series ring $\mathbb{Z}_p[\![x_1, \ldots, x_{n-1}]\!]$, that $\mathfrak{m} \subset R$ is its maximal ideal, and that $k \cong R/\mathfrak{m}$ is its residue field $\mathbb{F}_p$. The map $R \to k$ is Gorenstein, with associated Brown-Comenetz dualizing module $\mathcal{G}$. For a finite length (1.1) $R$-module $M$, the dual $D_{\mathcal{G}}(M)$ is given by

$$D_{\mathcal{G}}(M) \sim \Sigma^{-n}\,\mathrm{Ext}^n_R(M, R)\,.$$

Precisely as in 1.6, the $\mathbb{Z}_p$-module $\mathbb{Z}/p^\infty$ is a Brown-Comenetz dualizing module for $\mathbb{Z}_p \to \mathbb{F}_p$, and as in 1.8 there is a coinduced Brown-Comenetz dualizing module $\mathcal{I} = \mathcal{I}_0$ for $R \to k$. For $M$ as before the dual

$$D_{\mathcal{I}}(M) \sim \mathrm{Ext}^0_{\mathbb{Z}_p}(M, \mathbb{Z}/p^\infty)$$

is the ordinary Pontriagin dual of $M$. It turns out (3.10) that $\mathcal{G}$ is equivalent as an $R$-module to $\Sigma^{-n}\mathcal{I}$, and hence that on the category of

finite length $R$-modules, the functor $\mathrm{Ext}^n_R(-, R)$ is naturally isomorphic to $\mathrm{Ext}^0_{\mathbb{Z}_p}(-, \mathbb{Z}/p^\infty)$.

1.13. *Example (Poincaré duality).* (See [5] and [15].) This example is based on the following theorem.

1.14. **Proposition.** *Suppose that $X$ is a based finite 1-connected CW-complex, $k = \mathbb{S}$, and $R = C^*(X; k)$, as in 1.7. Then $R \to k$ is Gorenstein if and only if $X$ is a Poincaré duality space.*

In the situation of 1.14, there are usually many Brown-Comenetz dualizing modules for $R \to k$: these are exactly the Thom spectra $X^\rho$ obtained from stable spherical fibrations $\rho$ over $X$. If $X$ is a Poincaré duality space of formal dimension $d$, then as in [1] the Brown-Comenetz dualizing module $\mathcal{G} = \mathrm{Cell}_k(R) \sim R$ provided by the Gorenstein condition [5, 8.6] is $X^\nu$, where $\nu$ is the stable Spivak normal bundle of $X$, desuspended to have stable fibre dimension $-d$.

Since the spectrum $k$ is a Brown-Comenetz dualizing module for $k \to k$, it follows as in 1.8 there is a coinduced Brown-Comenetz dualizing module $\mathcal{I} = \mathcal{I}_0$ for $R \to k$ ([5, 9.16], 2.8). This coinduced dualizing module is the Thom complex $X^0$ of the trivial bundle. The $R$-module $\mathcal{G}$ is equivalent to $\mathcal{I}$ (up to suspension) if and only if $\nu$ is trivial, or in other words if and only if $X$ is orientable for stable cohomotopy.

Observe that by the Thom isomorphism theorem, the two dualizing modules $\mathcal{G}$ and $\mathcal{I}$ cannot be distinguished by mod 2 cohomology, although they can sometimes be distinguished by the action of the Steenrod algebra on mod 2 cohomology.

1.15. *Aside on functoriality.* For later purposes we describe an extended functoriality property of the isomorphisms described in 1.12. Let $R \to k$ be as in 1.12, but widen the module horizon to include the category of finite length (1.1) *skew* $R$-modules: the objects are ordinary $R$-modules as before, but a map $M \to M'$ is a pair $(\sigma, \tau)$, where $\sigma$ is an automorphism of $R$ and $\tau : M \to M'$ is a map of abelian groups such that for $r \in R$ and $m \in M$, $\tau(rm) = \sigma(r)\tau(m)$. Both $D_{\mathcal{G}}$ and $D_{\mathcal{I}}$ extend to this larger category (with the same definitions as before), but the functors are *not* naturally equivalent there. This is reflected in the fact that if $G = \mathrm{Aut}(R)$, then the twisted group ring $R[G]$ acts naturally both on $\mathcal{G}$ and on $\mathcal{I}$ in such a way that $\mathcal{G}$ and $\mathcal{I}$ are equivalent as $R$-modules, but not as $R[G]$-modules. The discrepancy between $\mathcal{G}$ and $\mathcal{I}$ has a simple description. Let $S = \mathbb{Z}_p[\![x_1, \ldots, x_{n-1}, y_1 \ldots, y_{n-1}]\!]$ be the evident completion of $R \oslash_{\mathbb{Z}_p} R$ and let $\mathcal{L} = \mathrm{Tor}^S_{n-1}(R, R)$. (The object $\mathcal{L}$ might be characterized as a type of Hochschild homology group of $R$.) Then $\mathcal{L}$ is an ordinary $R[G]$-module which is free of

rank 1 as an $R$-module, and there is a natural map $\Sigma^n \mathcal{L} \otimes_R \mathcal{G} \to \mathcal{I}$ of $R[G]$-modules which is an equivalence (3.10). (The action of $G$ on the tensor product is diagonal). This implies that on the category of finite length skew $R$-modules there is a natural isomorphism of functors

$$\mathcal{L} \oslash_R \operatorname{Ext}_n^R(-, R) \sim \operatorname{Ext}_0^{\mathbb{Z}_p}(-, \mathbb{Z}/p^\infty)\,.$$

1.16. **Gross-Hopkins duality.** Fix an integer $n \geq 1$, and let $L = L_n$ denote the localization functor on the stable category corresponding to the homology theory $K(n) \vee \cdots \vee K(0)$ , where $K(i)$ is the $i$'th Morava $K$-theory. Let $\mathcal{S} = L_n(\mathbb{S})$ and let $K = K(n)$. There is an essentially unique $\mathbb{S}$-algebra homomorphism $\mathcal{S} \to K$. The first component of Gross-Hopkins duality is the following statement.

1.17. **Theorem.** *The homomorphism $\mathcal{S} \to K$ is Gorenstein.*

This theorem provides a Brown-Comenetz dualizing module $\mathcal{G} = \operatorname{Cell}_k \mathcal{S}$ for $\mathcal{S} \to K$. The ordinary Brown-Comenetz dualizing spectrum $\mathbb{I}$ is a Brown-Comenetz dualizing module for $\mathbb{S} \to K$; as in 1.8 this gives rise to a coinduced Brown-Comenetz dualizing module $\mathcal{I} = \operatorname{Cell}_k \operatorname{Hom}_{\mathbb{S}}(\mathcal{S}, \mathbb{I})$ for $\mathcal{S} \to K$ (2.8, 2.12). The second component of Gross-Hopkins duality is the assertion that $\mathcal{G}$ *cannot* be distinguised from $\mathcal{I}$ by the most relevant applicable homological functor. This is analogous in this context to the Thom isomorphism theorem (cf. 1.13).

Let $E$ be the $\mathbb{S}$-algebra of [17], with

$$E_* = \pi_* E = W[\![u_1, \ldots, u_{n-1}]\!][u, u^{-1}]\,,$$

where $u_k$ is of degree 0, $u$ is of degree 2, and $W$ is the Witt ring of the finite field $\mathbb{F}_{p^n}$. For spectra $X$ and $Y$, let $X \hat{\otimes} Y = L_K(X \otimes Y)$, where $L_K$ is localization with respect to $K$. Following [17], for any $X$ we write $E_*^\vee(X) = \pi_*(E \hat{\otimes} X)$.

1.18. **Theorem.** *Both $E_*^\vee \mathcal{G}$ and $E_*^\vee \mathcal{I}$ are rank* 1 *free modules over $E_*$.*

The final and most difficult component of Gross-Hopkins duality is a determination of how $E_*^\vee \mathcal{G}$ differs from $E_*^\vee \mathcal{I}$ as a module over the ring of operations in $E_*$; this is analogous to distinguishing between two Thom complexes by considering the action of Steenrod algebra on mod 2 homology (cf. 1.13).

We begin by comparing the homologies of $D_{\mathcal{G}}(\mathcal{F})$ and $D_{\mathcal{I}}(\mathcal{F})$ when $\mathcal{F}$ is a finite complex of type $n$, i.e., a module over $\mathcal{S}$ which is finitely built from $\mathcal{S}$ and has $K(i)_*\mathcal{F} = 0$ for $i < n$ and $K(n)_*\mathcal{F} \neq 0$. These conditions imply that each element of $E_*^\vee \mathcal{F}$ is annihilated by some power of the maximal ideal $\mathfrak{m} \subset E_0$ [13, 8.5].

1.19. **Proposition.** *Suppose that $\mathcal{F}$ is a finite complex of type $n$. Then there are natural isomorphisms*

$$E^\vee_{-i}D_{\mathcal{G}}\mathcal{F} \;\cong\; \mathrm{Ext}^n_{E_0}(E^\vee_{i-n}\mathcal{F}, E_0)$$
$$E^\vee_{-i}D_{\mathcal{I}}\mathcal{F} \;\cong\; \mathrm{Ext}^0_{\mathbb{Z}_p}(E^\vee_{i+n^2}\mathcal{F}, \mathbb{Z}/p^\infty)\,.$$

Recall [17] that the Morava stabilizer group $\Gamma$, in one of its forms, is a profinite group of multiplicative automorphisms of $E$. The ring $\pi_*\operatorname{End}_{\mathcal{S}}(E)$ is the completed twisted group ring $E_*[\![\Gamma]\!]$ (see [17, pf. of Prop. 16]), and so, up to completion and multiplication by elements in $E_*$, the operations in $E_*$ are all of degree 0 and are determined by the action of elements of $\Gamma$. If $X$ is a spectrum, then $\Gamma$ acts on $E^\vee_*(X)$ as a group of automorphisms in the category of skew $E_*$-modules (1.15). It follows from naturality that the isomorphisms in 1.19 are $\Gamma$-equivariant, where, for instance, $\Gamma$ acts on $\mathrm{Ext}^n_{E_0}(E^\vee_{i-n}\mathcal{F}, E_0)$ in a diagonal way involving actions on all three constituents of the Ext. According to 1.12, the modules

$$\mathrm{Ext}^n_{E_0}(E^\vee_i\mathcal{F}, E_0) \text{ and } \mathrm{Ext}^0_{\mathbb{Z}_p}(E^\vee_i\mathcal{F}, E_0)$$

are isomorphic for any $i$; the question is to what extent these isomorphisms do or do not respect the action of $\Gamma$.

This is exactly the issue discussed in 1.15. Given 1.19 and 1.15, the following proposition is immediate (cf. 3.11). Let

$$T = W[\![u_1, \ldots, u_{n-1}, u'_1, \ldots, u'_{n-1}]\!]$$

be the evident completion of $E_0 \oslash_W E_0$, and let $\mathcal{L} = \mathrm{Tor}^T_{n-1}(E_0, E_0)$. It turns out (3.10) that $\mathcal{L}$ is a free module of rank 1 over $E_0$.

1.20. **Proposition.** *For any finite complex of type $n$, there are natural isomorphisms*

$$\mathcal{L} \otimes_{E_0} E^\vee_{i-n-n^2}D_{\mathcal{G}}X \cong E^\vee_i D_{\mathcal{I}}X$$

*of modules over $E_0[\![\Gamma]\!]$.*

1.21. *Remark.* We emphasize that in 1.20 the action of $\Gamma$ on the left-hand module is diagonal, and involves a nontrivial action of $\Gamma$ on $\mathcal{L}$.

This easily leads to the following proposition.

1.22. **Proposition.** *There are natural isomorphisms of $E_0[\![\Gamma]\!]$-modules*

$$\mathcal{L} \otimes_{E_0} E^\vee_{i-n-n^2}\mathcal{G} \cong E^\vee_i\mathcal{I}\,.$$

As in [17], there is a determinant-like map $\det : \Gamma \to \mathbb{Z}_p^\times$. If $M$ is an ordinary module over $E_0[\![\Gamma]\!]$ or $E_*[\![\Gamma]\!]$, write $M[\det]$ for the module obtained from $M$ by twisting the action of $\Gamma$ by det. The key computation made in [9] by Gross and Hopkins (which we do not rederive) involves the action of $\Gamma$ on $\mathcal{L}$.

1.23. **Theorem.** [10, Th. 6] *As a module over the twisted group ring* $E_0[\![\Gamma]\!]$, $\mathcal{L}$ *is isomorphic to* $E_{2n}[\det]$.

The first statement below follows from the fact that $\mathcal{G} \to \mathcal{S}$ is a $K_*$-equivalence (2.17) and hence an $E_*^\vee$-equivalence; the second is a combination of 1.22 and 1.23.

1.24. **Theorem.** [17] *There are isomorphisms of* $E_*[\![\Gamma]\!]$*-modules:*

$$\begin{aligned} E_*^\vee \mathcal{G} &\cong E_* \\ E_*^\vee \mathcal{I} &\cong \Sigma^{n^2-n} E_*[\det]\,. \end{aligned}$$

Finally, we give an analogue of the classification of Brown-Comenetz dualizing modules from 1.13. Recall that a $K$-local spectrum $M$ is said to be *invertible* if there is a $K$-local spectrum $N$ with $M \hat{\otimes} N \sim L_K(\mathcal{S})$.

1.25. **Theorem.** *Let* $\mathcal{E} = \operatorname{End}_{\mathcal{S}}(E)$. *Up to equivalence, there are bijective correspondences between the following three kinds of objects:*

(1) *invertible* $K$*-local spectra,*
(2) *Brown-Comenetz dualizing modules for* $\mathcal{S} \to K$ *,*
(3) *right actions of* $\mathcal{E}$ *on a suspension of* $E$ *which extend the natural right action of* $E$ *on itself.*

1.26. *Remark.* Suppose that $X$ is a based CW-complex, $G$ is the loop space on $X$ (constructed as a simplicial group), and $\mathcal{E} = \mathbb{S}[G]$ is the ring spectrum obtained as the unreduced suspension spectrum of $G$. Let $k = \mathbb{S}$ and $R = C^*(X; k)$ as in 1.13. Say that a module $M$ over $R$ is invertible if there is a module $N$ such that $M \otimes_R N \sim R$. Then if $X$ is finite and 1-connected, $\mathcal{E}$ is equivalent to $\operatorname{End}_R(k)$ [5], and 1.25 becomes in part analogous to the statement that up to equivalence there are bijective correspondences between the following four kinds of objects:

(1) invertible modules over $R$,
(2) Brown-Comenetz dualizing modules for $R \to k$,
(3) actions of $\mathcal{E}$ on a suspension of $k$ which (necessarily) extend the action of $k$ on itself, and
(4) stable spherical fibrations over $X$.

1.27. **Organization of the paper.** Section 2 has a short discussion of cellularity, §3 expands on some of the algebraic issues discussed in 1.15, and §4 recalls some material from stable homotopy theory. Section 5 contains the proofs of 1.17, 1.18, 1.19, and 1.22. The last section has a proof of 1.25.

1.28. **Notation and terminology.** If $R$ is a ring spectrum, we write ${}_R\mathrm{Mod}$ and $\mathrm{Mod}_R$ for the respective categories of left and right module spectra over $R$. If $X$ and $Y$ are spectra, $\mathrm{Hom}(X, Y)$ stands for $\mathrm{Hom}_{\mathbb{S}}(X, Y)$ and $X \otimes Y$ for $X \otimes_{\mathbb{S}} Y$. Note, though, that $\mathcal{S} \otimes_{\mathbb{S}} \mathcal{S} \sim \mathcal{S}$; this implies that if $X$ and $Y$ are $\mathcal{S}$-modules then $\mathrm{Hom}_{\mathcal{S}}(X, Y) \sim \mathrm{Hom}(X, Y)$ and $X \otimes_{\mathcal{S}} Y \sim X \otimes Y$. Whenever possible we use the simpler notation (without the subscript $\mathcal{S}$). If $X$ is a spectrum, $\hat{X} = L_K X$ stands for the $K$-localization of $X$; we also write $\hat{D}$ for $D_{\hat{\mathcal{S}}}$, so that $\hat{D}X = \mathrm{Hom}(X, \hat{\mathcal{S}})$.

For us, a *finite complex of type $n$* is an $\mathcal{S}$-module $\mathcal{F}$ which is finitely built from $\mathcal{S}$, such that $K(i)_*\mathcal{F} = 0$ for $i < n$ and $K(n)_*\mathcal{F} \neq 0$. If $F(n)$ is a finite complex of type $n$ in the sense of [13, §1.2], then $\mathcal{S} \otimes F(n)$ is a finite complex of type $n$ in our sense.

**Some technicalities.** The localized sphere $\mathcal{S}$ is a commutative $\mathbb{S}$-algebra, as is the spectrum $E$ [7, §7]. Disconcertingly, $K$ usually has an uncountable number of $\mathbb{S}$-algebra structures, none of them commutative; nevertheless we fix one $\mathbb{S}$-algebra structure on $K$ (nothing to follow will depend on this choice), and work with the essentially unique $\mathbb{S}$-algebra map $\mathcal{S} \to K$.

## 2. Cellularity and Koszul complexes

In this section we review the idea of cellularity, and look at how it fits in with the effective constructibility condition which appears in the definition of Brown-Comenetz dualizing module.

2.1. **Cellularity and cellular approximation.** Suppose that $R$ is an $\mathbb{S}$-algebra and that $k$ is an $R$-module. Recall that a subcategory of the category of $R$-modules is said to be *thick* if it is closed under (de)suspensions, equivalences, cofibration sequences, and retracts; it is *localizing* if in addition it is closed under arbitrary coproduts.

2.2. **Definition.** An $R$-module is *finitely built from $k$* if it belongs to the smallest thick subcategory of ${}_R\mathrm{Mod}$ which contains $k$. An $R$-module is *built from $k$* or is *$k$-cellular* if it belongs to the smallest localizing subcategory of ${}_R\mathrm{Mod}$ which contains $k$.

2.3. **Definition.** A map $f : M \to N$ of $R$-modules is a $\mathrm{Cell}_k$*-equivalence* if it induces an equivalence $\mathrm{Hom}_R(k, M) \sim \mathrm{Hom}_R(k, N)$.

It is not hard to see that a $\mathrm{Cell}_k$-equivalence between $k$-cellular $R$-modules is actually an equivalence; this follows for instance from the fact that a $\mathrm{Cell}_k$-equivalence $M \to N$ induces an equivalence $\mathrm{Hom}_R(C, M) \sim \mathrm{Hom}_R(C, N)$ for any $k$-cellular $C$. The main general result in this area is an approximation theorem. A map $M' \to M$ is

said to be a *$k$-cellular approximation* if $M'$ is $k$-cellular and $M' \to M$ is a $\mathrm{Cell}_k$-equivalence.

2.4. **Theorem.** [8, I.5] *Any $R$-module $M$ has a functorial $k$-cellular approximation $\mathrm{Cell}_k(M) \to M$. A map $M' \to M$ is a $\mathrm{Cell}_k$-equivalence if and only if the induced map $\mathrm{Cell}_k(M') \to \mathrm{Cell}_k(M)$ is an equivalence.*

2.5. *Constructing* $\mathrm{Cell}_k(M)$. In general, it is difficult to give a simple formula for $\mathrm{Cell}_k(M)$; the usual method for constructing it involves transfinite induction. But let $\mathcal{E} = \mathrm{End}_R(k)$ and note that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(k, \mathrm{Cell}_k M) \otimes_{\mathcal{E}} k & \longrightarrow & \mathrm{Cell}_k M \\ \sim\downarrow & & \downarrow \\ \mathrm{Hom}_R(k, M) \otimes_{\mathcal{E}} k & \longrightarrow & M \end{array}$$

in which the horizontal maps are evaluation. It is easy to conclude from this diagram that the following three conditions are equivalent:

(1) for all $M$, $\mathrm{Hom}_R(k, M) \otimes_{\mathcal{E}} k \to M$ is a $k$-cellular approximation,
(2) for all $M$, $\mathrm{Cell}_k M$ is effectively constructible from $k$ (1.2), and
(3) any $k$-cellular $R$-module is effectively constructible from $k$.

If these conditions hold, then the functor $\mathrm{Cell}_k(–)$ is easy to describe explicitly: it is equivalent to $\mathrm{Hom}_R(k, –) \otimes_{\mathcal{E}} k$.

We will next identify certain pairs $(R, k)$ for which the conditions of 2.5 are satisfied.

2.6. **Koszul complexes.** A *Koszul complex* for an $R$-module $k$ is an $R$-module $C$ which satisfies the following three conditions:

(1) $C$ is finitely built from $R$,
(2) $C$ is finitely built from $k$, and
(3) $C$ builds $k$.

If $R \to k$ is a map of $\mathbb{S}$-algebras, a *Koszul complex for $R \to k$* is a Koszul complex for $k$ as a left $R$-module.

2.7. **Proposition.** *Suppose that $R$ is an $\mathbb{S}$-algebra and $k$ is a module over $R$ which admits a Koszul complex $C$. Then the three conditions of 2.5 hold for $(R, k)$.*

*Proof.* Let $\mathcal{E} = \mathrm{End}_R(k)$. We will prove that if $M$ is any $R$-module, then the natural map $\lambda : \mathrm{Hom}_R(k, M) \otimes_{\mathcal{E}} k \to M$ is a $k$-cellular approximation. The domain of $\lambda$ is built from $k$ over $R$, because $\mathrm{Hom}_R(k, M)$ is built from $\mathcal{E}$ as a right module over $\mathcal{E}$, so it will be sufficient to

prove that $\lambda$ is a $\mathrm{Cell}_k$-equivalence. We look for $R$-modules $A$ with the property that the natural map

$$\mathrm{Hom}_R(k, M) \otimes_{\mathcal{E}} \mathrm{Hom}_R(A, k) \to \mathrm{Hom}_R(A, M)$$

is an equivalence. The module $A = k$ certainly works, and hence so does any module finitely built from $k$, e.g., the Koszul complex $C$. Since $C$ is finitely built from $R$, $\mathrm{Hom}_R(k, M) \otimes_{\mathcal{E}} \mathrm{Hom}_R(C, k)$ is equivalent to $\mathrm{Hom}_R(C, \mathrm{Hom}_R(k, M) \otimes_{\mathcal{E}} k)$. The conclusion is that the map $\lambda$ is a $\mathrm{Cell}_C$-equivalence. Since $C$ builds $k$, it follows that the map is also a $\mathrm{Cell}_k$-equivalence. □

In the presence of a Koszul complex, it is easier to recognize Gorenstein homomorphisms.

2.8. **Proposition.** [5, 8.4] *Suppose that $R \to k$ is a map of $\mathbb{S}$-algebras such that $k$, as an $R$-module, admits a Koszul complex. Then $R \to k$ is Gorenstein if and only if there is some integer $d$ such that* $\mathrm{Hom}_R(k, R)$ *is equivalent to $\Sigma^d k$ as a module over $k$.*

*Proof.* Since $\mathrm{Cell}_k(R)$ is effectively constructible from $k$ (2.7), the map $R \to k$ is Gorenstein if and only if there is some integer $d$ such that $\mathrm{Hom}_R(k, \mathrm{Cell}_k(R))$ is equivalent to $\Sigma^d k$ as a $k$-module. The proposition follows from the fact that the cellular approximation map $\mathrm{Cell}_k(R) \to R$ induces an equivalence on $\mathrm{Hom}_R(k, -)$. □

2.9. **Examples of Koszul complexes.** Suppose that $R$ is an ordinary commutative ring and that $k$ is a field which is a quotient of $R$ by a finitely generated ideal $\langle r_1, \ldots, r_m \rangle$. Let $C_i$ denote the complex $R \xrightarrow{r_i} R$ (concentrated in degrees 0 and $-1$), and $C$ the complex $C_1 \otimes_R \cdots \otimes_R C_m$. This is what is usually called the Koszul complex for $R \to k$; the following shows that definition 2.6 is consistent with this usage.

2.10. **Proposition.** [5, 3.2] *In the above situation, $C$ is a Koszul complex for $R \to k$ (in the sense of 2.6).*

Recall that $\hat{\mathcal{S}}$ is the localization of $\mathbb{S}$ with respect to the Morava $K$-theory $K(n)$. The unit map $\mathbb{S} \to E$ extends uniquely to an $\mathbb{S}$-algebra map $\hat{\mathcal{S}} \to E$.

2.11. **Proposition.** *The spectrum $\hat{\mathcal{S}}$ is a Koszul complex for $\hat{\mathcal{S}} \to E$.*

*Proof.* It follows from [13, 8.9, p. 48], that $\hat{\mathcal{S}}$ is finitely built from from $E$ (but don't ignore the notational discrepancy described in the proof of 4.3). It is clear that $\hat{\mathcal{S}}$ finitely builds itself, and, since $E$ is an $\hat{\mathcal{S}}$-module, that $\hat{\mathcal{S}}$ builds $E$. □

Let $\mathcal{F}$ be a fixed finite complex of type $n$ (1.28).

2.12. **Proposition.** *The $\mathcal{S}$-module $\mathcal{F}$ is a Koszul complex for $\mathcal{S} \to K$.*

*Proof.* By construction, $\mathcal{F}$ is finitely built from $\mathcal{S}$. Since $K \otimes \mathcal{F}$ is a nontrivial sum of copies of $K$, it is clear that $\mathcal{F}$ builds $K$. Finally, [13, 8.12] shows that $\mathcal{F}$ is finitely built from $K$. □

2.13. **Proposition.** *Let $\mathcal{E}$ denote the endomorphism spectrum* $\operatorname{End}(E)$. *Then $E$ is a Koszul complex for itself as a module over $\mathcal{E}$.*

*Proof.* As above, $\hat{\mathcal{S}}$ is finitely built from $E$. It follows immediately that $E = \operatorname{Hom}(\hat{\mathcal{S}}, E)$ is finitely built from $\mathcal{E} = \operatorname{Hom}(E, E)$ as a left module over $\mathcal{E}$. □

2.14. **Self-dual Koszul complexes.** Suppose that $R$ is a commutative $\mathbb{S}$-algebra. A module $M$ over $R$ is said to be *self-dual with respect to Spanier-Whitehead duality* if there is some integer $e$ such that $\operatorname{Hom}_R(M, R)$ is equivalent to $\Sigma^e M$ as an $R$-module. The following observation is less specialized than it seems.

2.15. **Proposition.** *Suppose that $R$ is a commutative $\mathbb{S}$-algebra, and that $k$ is an $R$-module which admits a Koszul complex with is self-dual with respect to Spanier-Whitehead duality. Then a map $f : M \to M'$ of $R$-modules is a* $\operatorname{Cell}_k$*-equivalence if and only if it induces an equivalence $k \otimes_R M \to k \otimes_R M'$.*

*Proof.* Let $C$ be the self-dual Koszul complex. Since $C$ and $k$ build one another, $f$ induces an equivalence on $\operatorname{Hom}_R(k, –)$ (i.e., is a $\operatorname{Cell}_k$-equivalence) if and only if it induces an equivalence on $\operatorname{Hom}_R(C, –)$. Moreover, $f$ induces an equivalence on $k \otimes_R –$ if and only if it induces an equivalence on $C \otimes_R –$. Since $C$ is finitely built from $R$, the functor $\operatorname{Hom}_R(C, –)$ is equivalent to $\operatorname{Hom}_R(C, R) \otimes_R –$. The proposition follows from the fact that $\operatorname{Hom}_R(C, R)$ is equivalent to $\Sigma^e C$. □

2.16. *Example.* The Koszul complex from 2.10 is self-dual; compare [4, 6.5].

2.17. *Example.* The Koszul complex $\mathcal{F}$ from 2.12 can be chosen to be self-dual; just replace $\mathcal{F}$ if necessary by $\mathcal{F} \otimes_{\mathcal{S}} D_{\mathcal{S}}\mathcal{F}$. It follows that a map of $\mathcal{S}$-modules is a $\operatorname{Cell}_K$-equivalence if and only if it induces an isomorphism on $K_*$. In particular, for any $\mathcal{S}$-module $X$ the map $X \to L_K X$ is a $\operatorname{Cell}_K$-equivalence and the map $\operatorname{Cell}_K X \to X$ is a $K_*$-equivalence.

## 3. Commutative Rings

In this section we will look at several examples of Gorenstein homomorphisms $R \to k$ between ordinary noetherian commutative rings. In each case $R$ is a regular ring, and $R \to k$ is projection to a residue field. In this situation $R \to k$ is Gorenstein [16], i.e., $\mathrm{Ext}^i_R(k, R)$ vanishes except in one degree, and in that degree is isomorphic to $k$ (2.8, 2.10). Indeed, to see this localize $R$ if necessary at the maximal ideal $\mathfrak{m} = \ker(R \to k)$ and observe that the Koszul complex on a minimal generating set for $\mathfrak{m}$ is a resolution of $k$. It is then clear from calculation that $\mathrm{Ext}^*_R(k, R)$ is isomorphic to (a shift of) $k$. There are three examples; the third is a combination of the first two. We give special attention to the extended functoriality issues discussed in 1.15. In this section we sketch arguments which explain where the results come from; these issues are treated in [14] from a very different point of view.

3.1. **$p$-adic number rings.** Let $R$ be the ring $\mathbb{Z}_p$ of $p$-adic integers, and $k$ the finite field $R/pR \cong \mathbb{Z}/p$. The ring $R$ is regular, hence $R \to k$ is Gorenstein and there is a Brown-Comenetz dualizing module $\mathcal{G} = \mathrm{Cell}_k(R)$ provided by the Gorenstein condition. If $M$ is a finitely generated $p$-primary torsion abelian group, the associated notion of duality is given by

$$D_{\mathcal{G}}(M) \sim \Sigma^{-1}\,\mathrm{Ext}^1_R(M, R)\,.$$

The Ext-group on the right is naturally isomorphic to the Pontriagin dual of $M$, and in fact the short exact sequence

$$0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Z}/p^\infty \to 0$$

can be used to produce an equivalence $\mathcal{G} \sim \Sigma^{-1}\mathbb{Z}/p^\infty$. All extended naturality issues (1.15) are trivial, if only because $R$ has no nontrivial automorphisms. In this case Gorenstein duality and Pontriagin duality coincide (up to suspension) on the category of finite length (1.1) skew $R$-modules.

A more interesting possibility is to let $R$ be the ring of integers in a finite unramified extension field of $\mathbb{Q}_p$, and $k$ the residue field of $R$. Again $R$ is regular, $R \to k$ is Gorenstein, and there is a Brown-Comenetz dualizing module $\mathcal{G} = \mathrm{Cell}_k(R)$ provided by the Gorenstein condition. However, as in 1.8 there is also a coinduced Brown-Comenetz dualizing module, given by $\mathcal{I} = \mathrm{Cell}_k \mathrm{Hom}_{\mathbb{Z}_p}(R, \mathbb{Z}/p^\infty)$. For an ordinary finitely-generated $p$-primary torsion $R$-module $M$, the two associated notions of duality are given by

$$\begin{aligned} D_{\mathcal{G}}(M) &\sim \Sigma^{-1}\,\mathrm{Ext}^1_R(M, R) \\ D_{\mathcal{I}}(M) &\sim \qquad \mathrm{Ext}^0_{\mathbb{Z}_p}(M, \mathbb{Z}/p^\infty)\,, \end{aligned}$$

where as before the lower Ext-group is the Pontriagin dual of $M$. Perhaps surprisingly, the two Ext-functors on the right are naturally isomorphic on the category of finite length skew $R$-modules. This can be proved by showing that there are equivalences

$$\begin{aligned} \mathcal{G} &\sim \Sigma^{-1}\mathbb{Z}/p^\infty \oslash_{\mathbb{Z}_p} R \\ \mathcal{I} &\sim \qquad \operatorname{Ext}^0_{\mathbb{Z}_p}(R, \mathbb{Z}/p^\infty) \end{aligned}$$

and observing that there is a canonical isomorphism

$$R \to \operatorname{Ext}^0_{\mathbb{Z}_p}(R, \mathbb{Z}_p)$$

given by the map which sends $r \in R$ to the trace over $\mathbb{Z}_p$ of the map $x \mapsto rx$. These considerations produce an $R[\operatorname{Aut}(R)]$-equivalence $\Sigma\mathcal{G} \sim \mathcal{I}$. Hence in this case, also, Gorenstein duality agrees up to suspension with Pontriagin duality as strongly as we might hope.

3.2. **Power series over a field.** Suppose that $R$ is the power series ring $k[\![x_1, \ldots, x_{n-1}]\!]$, and that $R \to k$ is the natural map sending $x_i$ to zero. The ring $R$ is regular, $R \to k$ is Gorenstein, and the Gorenstein condition provides a Brown-Comenetz dualizing module $\mathcal{G} = \operatorname{Cell}_k(R)$. As in 1.8, there is a coinduced Brown-Comenetz dualizing module $\mathcal{I} = \operatorname{Cell}_k \operatorname{Ext}^0_k(R, k)$. Let $\mathfrak{m}$ denote the kernel of $R \to k$. For a finite length (1.1) $R$-module $M$, the two associated notions of duality are given by

$$\begin{aligned} D_{\mathcal{G}}(M) &\sim \Sigma^{-(n-1)} \operatorname{Ext}^{n-1}_R(M, R) \\ D_{\mathcal{I}}(M) &\sim \qquad\qquad \operatorname{Ext}^0_k(M, k) \end{aligned} \tag{3.3}$$

Let $S'$ denote $R \oslash_k R \cong k[\![x_1 \ldots, x_{n-1}]\!] \oslash_k k[\![y_1, \ldots, y_{n-1}]\!]$, let $S = k[\![x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}]\!]$ be the evident completion of $S'$, and let $\mathcal{L}$ be given by the formula

$$\mathcal{L} = \pi_{n-1}(R \otimes_S R) \cong \operatorname{Tor}^S_{n-1}(R, R)\,.$$

(Here $R$ is treated as an $S$-module by the completed multiplication map $S \to R$ which has $x_i \mapsto x_i$ and $y_i \mapsto x_i$.) Note that $\operatorname{Aut}(R)$ acts naturally on $\mathcal{L}$. The following proposition compares the two dualities of 3.3.

3.4. **Proposition.** *The object $\mathcal{L}$ is a free (ordinary) R-module on one generator. For any finite length R-module $M$, there is an isomorphism*

$$\mathcal{L} \oslash_R \operatorname{Ext}^{n-1}_R(M, R) \cong \operatorname{Ext}^0_k(M, k)\,.$$

*which is natural with respect to skew homomorphisms $M \to M'$.*

3.5. *Remark.* Underlying 3.4 is an $R[\operatorname{Aut}(R)]$-equivalence

$$\Sigma^{n-1}\mathcal{L} \otimes_R \mathcal{G} \sim \mathcal{I}\,.$$

or an equivalence $\Sigma^{n-1}\mathcal{L} \sim \mathrm{Hom}_R(\mathcal{G}, \mathcal{I})$. On the indicated category of skew $R$-modules, Gorenstein duality agrees naturally (up to suspension) with Kronecker duality over $k$ only after twisting by $\mathcal{L}$.

Let $R_\epsilon$ denote $R$ considered as an ordinary $S$-module via the map $S \to R$ with $x_i \mapsto 0$ and $y_i \mapsto x_i$.

3.6. **Lemma.** *The natural map $R \otimes_{S'} R_\epsilon \to R \otimes_S R_\epsilon$ is an equivalence.*

*Proof.* This follows from an explicit calculation depending on the fact that for both $S$ and $S'$, the module $R_\epsilon$ is the quotient of the ring by the ideal generated by the regular sequence $(x_1, \ldots, x_{n-1})$. □

In the following lemma, $S$ acts on $\mathrm{Hom}_k(M, R) \sim \mathrm{Ext}^0_k(M, R)$ in a completed bimodule fashion, e.g., $(x_i \cdot f)(m) = f(x_i m)$ and $(y_i \cdot f)(m) = y_i f(m)$.

3.7. **Lemma.** *If $M$ is a finite length $R$-module, then the natural maps*

$$\mathrm{Hom}_S(R, \mathrm{Hom}_k(M, R)) \to \mathrm{Hom}_{S'}(R, \mathrm{Hom}_k(M, R))$$
$$R \otimes_{S'} \mathrm{Hom}_k(M, R) \to R \otimes_S \mathrm{Hom}_k(M, R)$$

*are equivalences.*

*Proof.* The module $M$ has a composition series in which the successive quotients are isomorphic to $k$; by an inductive argument, it suffices to treat the case $M = k$. In this case the second statement is 3.6, while the first follows from 3.6 and the equivalences

$$\mathrm{Hom}_S(R, R_\epsilon) \sim \mathrm{Hom}_R(R_\epsilon \otimes_S R, R_\epsilon)$$
$$\mathrm{Hom}_{S'}(R, R_\epsilon) \sim \mathrm{Hom}_R(R_\epsilon \otimes_{S'} R, R_\epsilon)\,.$$ □

We will use the fact that for any $R$-modules $A$ and $B$, there are natural weak equivalences

$$\begin{aligned} \mathrm{Hom}_R(A, B) &\sim \mathrm{Hom}_{S'}(R, \mathrm{Hom}_k(A, B)) \\ A \otimes_R B &\sim R \otimes_{S'} (A \otimes_k B) \end{aligned} \tag{3.8}$$

*Proof of 3.4 (sketch).* The fact that $\mathcal{L}$ is a free module of rank 1 over $R$ follows from calculation with the usual Koszul resolution of $R$ over $S$. Let $\mathcal{L}^\#$ denote $\mathrm{Ext}^{n-1}_S(R, S)$. Another calculation with the Koszul resolution shows that $\mathcal{L}^\#$ is also a free module of rank 1 over $R$, and that the composition pairing

$$\mathcal{L} \oslash_R \mathcal{L}^\# = \mathrm{Tor}^S_{n-1}(R, R) \oslash_R \mathrm{Ext}^{n-1}_S(R, S) \to \mathrm{Tor}^S_0(R, S) \cong R$$

is an isomorphism; see [2, Lemma 1.5]. To finish the proof, it is enough to show that for any $M$ as described there is a natural isomorphism

$$\mathcal{L}^\# \oslash_R \mathrm{Ext}^0_k(M, k) \to \mathrm{Ext}^{n-1}_R(M, R)\,.$$

Again, consideration of the Koszul resolution shows that $R$ is finitely built from $S$ as an $S$-module, and that $\mathrm{Ext}^i_S(R,S)$ vanishes if $i \neq n-1$. It follows that $\mathrm{Hom}_S(R,S)$ is equivalent to $\Sigma^{1-n}\mathcal{L}^{\#}$, and that for any $S$-module $X$ there is a natural isomorphism

$$\Sigma^{1-n}\mathcal{L}^{\#} \otimes_S X \sim \mathrm{Hom}_S(R,S) \otimes_S X \sim \mathrm{Hom}_S(R,X)\,. \tag{3.9}$$

Now let $M$ be a finite length $R$-module, and let $X$ be the ordinary $S$-module $\mathrm{Hom}_k(M,R)$. The module $M$ is finite-dimensional over $k$, and so $X$ is equivalent to $\mathrm{Hom}_k(M,k) \otimes_k R$, and (cf. 3.7, 3.8) 3.9 gives an equivalence

$$\begin{aligned}\Sigma^{1-n}\mathcal{L}^{\#} \otimes_R \mathrm{Hom}_k(M,k) &\sim \mathrm{Hom}_S(R,S) \otimes_R (R \otimes_S (\mathrm{Hom}_k(M,k) \otimes_k R))\\ &\sim \mathrm{Hom}_S(R,S) \otimes_S \mathrm{Hom}_k(M,R)\\ &\sim \mathrm{Hom}_S(R, \mathrm{Hom}_k(M,R))\\ &\sim \mathrm{Hom}_R(M,R)\,.\end{aligned}$$

Applying $\pi_{1-n}$ gives the desired isomorphism. Its construction is natural enough to respect skew homormorphisms $M \to M'$. □

3.10. **Power series over a $p$-adic ring.** Let $W$ be the ring of integers in a finite unramified extension field of $\mathbb{Q}_p$, $k$ the residue field of $W$, $R$ the formal power series ring $W[\![x_1,\ldots,x_{n-1}]\!]$, and $R \to k$ the quotient map sending each $x_i$ to zero. As before, $R \to k$ is Gorenstein and the Gorenstein condition provides a Brown-Comenetz dualizing module $\mathcal{G} = \mathrm{Cell}_k(R)$. As in 1.8, there is a coinduced Brown-Comenetz dualizing module $\mathcal{I} = \mathrm{Cell}_k \mathrm{Hom}_{\mathbb{Z}_p}(R, \mathbb{Z}/p^\infty)$. Let $\mathfrak{m}$ denote the kernel of $R \to k$. For a finite length (1.1) $R$-module $M$, the two associated notions of duality are given by

$$\begin{aligned} D_{\mathcal{G}}(M) &\sim \Sigma^{-n}\,\mathrm{Ext}^n_R(M,R)\\ D_{\mathcal{I}}(M) &\sim \qquad \mathrm{Ext}^0_{\mathbb{Z}_p}(M, \mathbb{Z}/p^\infty)\end{aligned}$$

Let $S = W[\![x_1,\ldots,x_{n-1},y_1,\ldots,y_{n-1}]\!]$ be the evident completion of $R \oslash_W R$, and let $\mathcal{L} = \pi_{n-1}(R \otimes_S R)$.

3.11. **Proposition.** *The object $\mathcal{L}$ is a free ordinary $R$-module on one generator. For any finite length $R$-module $M$, there is an isomorphism*

$$\mathcal{L} \oslash_R \mathrm{Ext}^n_R(M,R) \cong \mathrm{Ext}^0_{\mathbb{Z}_p}(M, \mathbb{Z}/p^\infty)\,.$$

*which is natural with respect to skew homomorphisms $M \to M'$.*

3.12. *Remark.* Behind this proposition is an equivalence $\Sigma^n\mathcal{L} \otimes_R \mathcal{G} \sim \mathcal{I}$. On the indicated category of skew $R$-modules, Gorenstein duality agrees naturally (up to suspension) with Pontriagin duality only after twisting by $\mathcal{L}$.

*Proof of 3.11 (sketch).* Let $\mathcal{L}^\#$ denote $\operatorname{Ext}_S^{n-1}(R,S)$. As in the proof of 3.4, it is enough to show that for all $M$ of the indicated type there is a natural isomorphism

$$\mathcal{L}^\# \oslash_R \operatorname{Ext}^0_{\mathbb{Z}_p}(M,\mathbb{Z}/p^\infty) \cong \operatorname{Ext}^n_R(M,R)\,.$$

Let $\mathfrak{n}\subset\mathfrak{m}$ denote the kernel of the map $R\to W$ sending each $x_i$ to zero. The arguments in the proof of 3.4 give an equivalence

$$\mathcal{L}^\# \otimes_R \operatorname{Hom}_W(M,W) \sim \Sigma^{n-1}\operatorname{Hom}_R(M,R)$$

for any ordinary finitely-generated $R$-module $M$ which is annihilated by a power of $\mathfrak{n}$. (The observation in the proof of 3.7 that $M$ has a composition series in which the successive quotients are isomorphic to $k$ has to be replaced by the observation that $M$ has a composition series in which the successive quotients are isomorphic as $R$-modules to cyclic modules over the PID $W$.) If in addition $M$ is $\mathfrak{m}$-primary, i.e., if $M$ is a $p$-primary torsion abelian group, then the considerations of 3.1 give an equivalence

$$\Sigma\operatorname{Hom}_W(M,W)\sim\operatorname{Hom}_{\mathbb{Z}_p}(M,\mathbb{Z}/p^\infty)\,.$$

Combining the equivalences, applying $\pi_*$, and verifying naturality gives the result. $\square$

## 4. Chromatic ingredients

The purpose of this section is to recall some material from [13] and [17]. As in 1.16, let $\mathcal{I}$ denote $\operatorname{Cell}^{\mathcal{S}}_K\operatorname{Hom}(\mathcal{S},\mathbb{I})$, where $\mathbb{I}$ is the ordinary Brown-Comenetz dual of the sphere.

4.1. *Remark.* Note that if $X$ is an $\mathcal{S}$-module which is built from $K$, then $\operatorname{Hom}(X,\mathcal{I})\sim\operatorname{Hom}(X,\operatorname{Hom}(\mathcal{S},\mathbb{I}))$ is equivalent to $\operatorname{Hom}(\mathcal{S}\otimes X,\mathbb{I})\sim\operatorname{Hom}(X,\mathbb{I})$. In particular, for such an $X$ the homotopy groups of $D_{\mathcal{I}}X$ are the Pontriagin duals of the homotopy groups of $X$.

4.2. **Proposition.** *The $\mathcal{S}$-module $\mathcal{I}$ is a Brown-Comenetz dualizing module for $\mathcal{S}\to K$.*

*Proof.* Given 4.1, a homotopy group calculation shows that $\operatorname{Hom}(K,\mathbb{I})$ is equivalent to $K$ as a left $K$-module. Since $\mathcal{S}\to K$ has a Koszul complex (2.12), the result follows from 2.8. $\square$

Recall that a $K$-local spectrum $X$ is said to be *invertible* if there exists a $K$-local spectrum $Y$ such that $X\hat{\otimes}Y\sim\hat{\mathcal{S}}$. In the following statement "shifted isomorphic" means "isomorphic up to suspension".

4.3. **Proposition.** *Suppose that $X$ is a $K$-local spectrum. Then the following conditions are equivalent:*

(1) *$X$ is invertible.*
(2) *$K_*X$ is shifted isomorphic to $K_*$ as a $K_*$-module.*
(3) *$K^*X$ is shifted isomorphic to $K^*$ as a $K^*$-module.*
(4) *$E^\vee_*X$ is shifted isomorphic to $E_*$ as an $E_*$-module.*
(5) *$E^*X$ is shifted isomorphic to $E^*$ as an $E^*$-module.*

*Proof.* This is essentially [13, 14.2]. There is a technical point to take into account. Hovey and Strickland use the letter "$E$" to denote a spectrum which we will call $\epsilon$; its homotopy groups are given by

$$\epsilon_* = \mathbb{Z}_p[\![v_1, v_2, \dots, v_{n-1}]\!][v_n, v_n^{-1}]\,.$$

where $|v_k| = 2(p^k - 1)$. Our ring $E_*$ is a finitely generated free module over $\epsilon_*$ under the map sending $v_k$ to $u^{p^{k-1}}u_k$, where $u_0 = p$, $u_n = 1$. Let $\epsilon^\vee_*(X)$ denote $\pi_*L_K(\epsilon \otimes X)$. Given the way in which the cohomology theories $\epsilon_*$ and $E_*$ are defined (i.e., by Landweber exactness [13, p. 4] [17]), for any spectrum $X$ there are isomorphisms

$$\begin{aligned} E_*(X) &\cong E_* \oslash_{\epsilon_*} \epsilon_*(X) \\ E^\vee_*(X) &\cong E_* \oslash_{\epsilon_*} \epsilon^\vee_*(X)\,. \end{aligned} \tag{4.4}$$

Hovey and Strickland show that conditions (1) and (2) and (3) of the proposition hold if and only if $\epsilon^\vee_*(X)$ is isomorphic to $\epsilon_*$ (up to suspension). The proof is completed by observing that, in view of 4.4, $E^\vee_*(X)$ is isomorphic to $E_*$ (up to suspension) if and only if $\epsilon^\vee_*(X)$ is equivalent to $\epsilon_*$ (up to suspension). Similar considerations apply to $E^*$. □

4.5. **Proposition.** *The $K$-local spectrum $\hat{\mathcal{I}}$ is invertible.*

*Proof.* This is [13, 10.2(e)]; see also Theorem 6.1. □

4.6. **Proposition.** *If $I$ is an invertible $K$-local spectrum, then the functor $X \mapsto X\hat{\otimes}I$ gives a self-equivalence of the homotopy category of $K$-local spectra. In particular, for any $K$-local spectra $X$, $Y$ the natural map*

$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}(X\hat{\otimes}I, Y\hat{\otimes}I)$$

*is an equivalence.*

*Proof.* The inverse functor is given by $X \mapsto X\hat{\otimes}J$, where $I\hat{\otimes}J \sim \hat{\mathcal{S}}$. □

4.7. *Remark.* If $I$ is invertible, the "multiplicative inverse" $J$ of $I$ is given by $J = \mathrm{Hom}(I, \hat{\mathcal{S}})$. This can be derived from the chain of equivalences

$$J \sim \mathrm{Hom}(\hat{\mathcal{S}}, J) \sim \mathrm{Hom}(I\hat{\otimes}\hat{\mathcal{S}}, I\hat{\otimes}J) \sim \mathrm{Hom}(I, \hat{\mathcal{S}})\,.$$

4.8. **Proposition.** [13, 10.6] *If $I$ is an invertible $K$-local spectrum, then for any spectrum $X$, the natural map* $\operatorname{Hom}(X, \hat{\mathcal{S}})\hat{\otimes} I \to \operatorname{Hom}(X, I)$ *is an equivalence.*

*Proof.* Pick a $K$-local $J$ such that $I\hat{\otimes}J \sim \hat{\mathcal{S}}$. Now use 4.6 to compute

$$\begin{aligned}\operatorname{Hom}(X, I) &\sim \operatorname{Hom}(X\hat{\otimes}J, I\hat{\otimes}J)\\ &\sim \operatorname{Hom}(J, \operatorname{Hom}(X, \hat{\mathcal{S}}))\\ &\sim \operatorname{Hom}(J\hat{\otimes}I, \operatorname{Hom}(X, \hat{\mathcal{S}})\hat{\otimes}I)\end{aligned}$$

and note that the final spectrum is $\operatorname{Hom}(X, \hat{\mathcal{S}})\hat{\otimes}I$. □

4.9. **Theorem.** [17, Prop. 16] *There is a weak equivalence*

$$\hat{D}E \sim \Sigma^{-n^2} E \tag{4.10}$$

*of left $E$-modules, which respects the actions of $\Gamma$ on both sides.*

*Proof.* Much of the content of this proof is in the technical details, but we will sketch the argument. Let $\mathcal{E} = \operatorname{End}(E)$. Note that the natural map

$$\operatorname{Hom}(E, X) \to \operatorname{Hom}_{\mathcal{E}}(\operatorname{Hom}(X, E), \operatorname{Hom}(E, E)) \tag{4.11}$$

is a weak equivalence for $X = E$. Since $\hat{\mathcal{S}}$ is finitely built from $E$ (2.11) and both sides of 4.11 respect cofibration sequences in $X$, it follows that 4.11 is an equivalence for $X = \hat{\mathcal{S}}$. This results in a strongly convergent Adams spectral sequence

$$E^2_{*,*} = \operatorname{Ext}^*_{E_*[\![\Gamma]\!]}(E_*, E_*[\![\Gamma]\!]) \Rightarrow \pi_* \operatorname{Hom}(E, \hat{\mathcal{S}})\,.$$

By a change of rings, the $E^2$-page is isomorphic to the continuous cohomology $H^*_c(\Gamma, E_*[\![\Gamma]\!])$. Since $\Gamma$ is a Poincaré duality group of dimension $n^2$, this continuous cohomology vanishes except in homological degree $n^2$, where it is isomorphic to $E_*$ [17, Prop. 5]. □

4.12. *Remark.* It follows from 4.9 that the natural map $\rho : E \to \hat{D}^2 E$ is an equivalence. To see this, let $f : \Sigma^{-n^2} E \to \hat{\mathcal{S}}$ be a map which corresponds under 4.9 to the unit in $E_0$. The adjoint of the equivalence $\Sigma^{-n^2} E \to \operatorname{Hom}(E, \hat{\mathcal{S}})$ is then the composite

$$(\Sigma^{-n^2} E) \otimes E \xrightarrow{m} \Sigma^{-n^2} E \xrightarrow{f} \hat{\mathcal{S}}\,, \tag{4.13}$$

where $m$ is obtained from the multiplication on $E$. Consider the following two maps

$$\rho, \lambda : E \to \hat{D}^2 E \sim \operatorname{Hom}(\Sigma^{-n^2} E, \hat{\mathcal{S}})\,,$$

which we will specify by giving their adjoints $E \otimes \Sigma^{-n^2} E \to \hat{\mathcal{S}}$. The adjoint of $\rho$ is the composite of 4.13 with the transposition map $E \otimes (\Sigma^{-n^2} E) \to (\Sigma^{-n^2} E) \otimes E$; the adjoint of $\lambda$ is obtained by shifting the suspension coordinate in 4.13 from one tensor factor to the other. The map $\lambda$ is an equivalence because the adjoint of 4.13 is an equivalence. The fact that $\rho$ is an equivalence now follows from the fact that $E$ is a commutative $\mathbb{S}$-algebra.

## 5. Gross-Hopkins duality

In this section we prove the main statements involved in Gross-Hopkins duality, except, of course, for the Gross-Hopkins calculation itself (1.23). We rely heavily on [13] and [17].

*Proof of 1.17.* By 2.17, $\mathrm{Hom}(K, \mathcal{S})$ is equivalent to $\mathrm{Hom}(K, \hat{\mathcal{S}})$. Use 4.5 and 4.6 to obtain an equivalence $\mathrm{Hom}(K, \hat{\mathcal{S}}) \sim \mathrm{Hom}(K \hat{\otimes} \hat{\mathcal{I}}, \hat{\mathcal{I}})$, observe (4.3) that $K \hat{\otimes} \hat{\mathcal{I}}$ is equivalent to $K$, and invoke 4.2 to evaluate $\mathrm{Hom}(K, \hat{\mathcal{I}}) \sim \mathrm{Hom}(K, \mathcal{I})$.

*Proof of 1.18.* For the statment involving $\mathcal{G}$, note that the map $\mathcal{G} \to \mathcal{S}$ is a $\mathrm{Cell}_K$-equivalence, and hence (2.17) an equivalence on $K_*$ or $E^\vee_*$. It follows that $E^\vee_* \mathcal{G}$ is isomorphic to $E^\vee_* \mathcal{S} \cong E_*$, even as modules over $\Gamma$. The statement involving $\mathcal{I}$ is a consequence of 4.5 and 4.3, since the localization map $\mathcal{I} \to \hat{\mathcal{I}}$ induces an isomorphism on $K_*$ or $E^\vee_*$. □

*Proof of 1.19.* For the first isomorphism, observe that because $\mathcal{F}$ is finitely built from $K$ [13, 8.12] and $\mathcal{G} \to \mathcal{S}$ is a $\mathrm{Cell}_K$-equivalence, $D_{\mathcal{G}}(\mathcal{F})$ is equivalent to $D_{\mathcal{S}}(\mathcal{F})$. Since $\mathcal{F}$ is finitely built from $\mathcal{S}$, the usual properties of Spanier-Whitehead duality give an equivalence

$$E \otimes D_{\mathcal{S}}(\mathcal{F}) \sim \mathrm{Hom}(\mathcal{F}, E)\,.$$

It follows from 1.17 that $D_{\mathcal{S}}(\mathcal{F})$ is also finitely built from $K$, which implies that $E \otimes D_{\mathcal{S}}(\mathcal{F})$ is $K$-local and hence equivalent to $E \hat{\otimes} D_{\mathcal{S}}(\mathcal{F})$. Combining these obervations gives an equivalence

$$E \hat{\otimes} D_{\mathcal{G}}(\mathcal{F}) \sim \mathrm{Hom}(\mathcal{F}, E)\,,$$

so that $E^\vee_i D_{\mathcal{G}}(\mathcal{F})$ is isomorphic to $E^{-i}(\mathcal{F})$. There is a strongly convergent universal coefficient spectral sequence

$$\mathrm{Ext}^*_{E_*}(E_* \mathcal{F}, E_*) \Rightarrow E^*(\mathcal{F})\,.$$

Since $E_*$ is isomorphic as a graded $E_*$-module to $\mathrm{Ext}^0_{E_0}(E_*, E_0)$, a standard change of rings argument (Shapiro's lemma) produces a spectral sequence

$$\mathrm{Ext}^i_{E_0}(E_j \mathcal{F}, E_0) \Rightarrow E_{-j-i} D_{\mathcal{G}}(\mathcal{F})\,.$$

But $E_0 \to \mathbb{F}_{p^n}$ is Gorenstein and each group $E_j\mathcal{F}$ has a finite composition series in which the successive quotients are isomorphic, as $E_0$-modules, to $\mathbb{F}_{p^n}$. This implies that the above Ext-groups vanish except for $i = n$, which leads to the desired result.

For the second isomorphism, observe that that there are equivalences

$$\begin{aligned} \operatorname{Hom}(E\hat{\otimes}\mathcal{F}, \hat{\mathcal{I}}) &\sim \operatorname{Hom}(\mathcal{F}, \operatorname{Hom}(E, \hat{\mathcal{I}})) \\ &\sim \operatorname{Hom}(\mathcal{F}, \Sigma^{-n^2} E\hat{\otimes}\hat{\mathcal{I}}) \\ &\sim \Sigma^{-n^2} E\hat{\otimes} \operatorname{Hom}(\mathcal{F}, \hat{\mathcal{I}}) \end{aligned} \tag{5.1}$$

where the second equivalence comes from combining 4.9 with 4.8, and the third from the fact that $\mathcal{F}$ is finite. Since $\mathcal{F}$ is finitely built out of $K$, $E\hat{\otimes}\mathcal{F} \sim E \otimes \mathcal{F}$ is built out of $K$, and the homotopy groups of the initial spectrum in the chain 5.1 are the Pontriagin duals of $E_*^\vee\mathcal{F}$ (4.1). The proof is completed by noting that the homotopy groups of the terminal spectrum in 5.1 are given by $E_*^\vee D_{\mathcal{I}}\mathcal{F}$. $\square$

*Proof of 1.22.* It follows from 4.5, 4.8 and the argument in the proof of 1.19 that for any finite complex $\mathcal{F}$ of type $n$ there is are equivalences $D_{\mathcal{G}}(\mathcal{F}) \sim D_{\mathcal{S}}(\mathcal{F}) \otimes \mathcal{G}$ and $D_{\mathcal{I}}(\mathcal{F}) \sim D_{\mathcal{S}}(\mathcal{F}) \otimes \mathcal{I}$. This gives Kunneth isomorphisms of modules over $E[\![\Gamma]\!]$:

$$\begin{aligned} E_*^\vee D_{\mathcal{I}}(\mathcal{F}) &\cong E_*^\vee D_{\mathcal{S}}(\mathcal{F}) \oslash_{E_*} E_*^\vee \mathcal{G} \\ E_*^\vee D_{\mathcal{I}}(\mathcal{F}) &\cong E_*^\vee D_{\mathcal{S}}(\mathcal{F}) \oslash_{E_*} E_*^\vee \mathcal{I} \end{aligned}$$

Let $\epsilon$ be the spectrum described in the proof of 4.3. Call an ideal $J \subset \epsilon_*$ *admissible* if it has the form $(p^{a_0}, v_1^{a_1}, \ldots, v_{n-1}^{a_{n-1}})$. As described in [13, §4], there exists a family $\{J_\alpha\}$ of admissible ideals, such that $\cap_k J_\alpha = 0$, and such that for each $\alpha$ there exists a finite complex $\mathcal{F}_\alpha$ of type $n$ with $\epsilon_*\mathcal{F}_\alpha \cong \epsilon_*/J_\alpha$. Under the inclusion $\epsilon_* \to E_*$ we can treat $J_\alpha$ as an ideal of $E_*$ and obtain (4.4) $E_*^\vee Y_\alpha \cong E_*/J_\alpha$. Let $X_\alpha = D_{\mathcal{S}}\mathcal{F}_\alpha$, so that $\mathcal{F}_\alpha \sim D_{\mathcal{S}} X_\alpha$. Then there are isomorphisms

$$\begin{aligned} E^\vee D_{\mathcal{G}} X_\alpha &\cong E_*^\vee(\mathcal{G})/J_\alpha \\ E^\vee D_{\mathcal{I}} X_\alpha &\cong E_*^\vee(\mathcal{I})/J_\alpha \,. \end{aligned}$$

The proof is completed by combining these isomorphisms with 1.19 and passing to the limit in $J_\alpha$ [13, 4.22].

## 6. Invertible modules

In this we prove 1.25. We begin with an extension of 4.3.

6.1. **Theorem.** *A $K$-local spectrum $I$ is invertible if and only if, up to suspension,* $\operatorname{Hom}(K, I)$ *is equivalent to $K$.*

6.2. *Remark.* It is easy to see that $\mathrm{Hom}(K, I)$ is equivalent to $\Sigma^d K$ as a spectrum if and only if it is equivalent to $\Sigma^d K$ as a $K$-module.

6.3. **Lemma.** *If $Y$ is $K$-local and $X$ is any spectrum, then* $\mathrm{Hom}(X, Y)$ *is $K$-local.*

*Proof.* One needs to show that if $A$ is $K$-acyclic, $\mathrm{Hom}(A, \mathrm{Hom}(X, Y))$ is contractible. But this spectrum is equivalent to $\mathrm{Hom}(X, \mathrm{Hom}(A, Y))$, and $\mathrm{Hom}(A, Y)$ is contractible because $Y$ is $K$-local. □

6.4. **Lemma.** *Suppose that $I$ is a $K$-local spectrum such that* $\mathrm{Hom}(K, I)$ *is equivalent to a suspension of $K$. Then the natural map $\kappa_X : X \to D_I^2(X)$ is an equivalence for $X = K$ and $X = \hat{\mathcal{S}}$.*

*Proof.* We can shift $I$ by a suspension and assume $\mathrm{Hom}(K, I) \sim K$. Let $f : K \to I$ be essential. Under the identification $K \sim \mathrm{Hom}(K, I)$ obtained by choosing $f$ as a generator for $\pi_* \mathrm{Hom}(K, I)$ as a module over $K_*$, the map $\kappa_K$ is adjoint to the composite of $f$ with the multiplication map $K \otimes K \to K$. Since $\mathrm{Hom}(K, I)$ is clearly equivalent to $K$ both as a left module and as a right module over $K$, it is easy to conclude that $\kappa_K$ is an equivalence (cf. 4.12).

By a thick subcategory argument, $\kappa_X$ is an equivalence for all spectra finitely built from $K$, e.g., for a finite spectrum $\mathcal{F}$ of type $n$. Since $D_I(\mathcal{F}) \sim D_{\mathcal{S}}(\mathcal{F}) \otimes I$ and $\mathcal{F} \sim D_{\mathcal{S}}^2(\mathcal{F})$, the spectrum $D_I^2(\mathcal{F})$ can be identified with $\mathcal{F} \otimes \mathrm{Hom}(I, I)$. It follows that

$$K_*(\mathcal{F}) \cong K_*(\mathcal{F}) \oslash_{K_*} K_* \mathrm{Hom}(I, I)$$

and hence that $K_* \mathrm{Hom}(I, I) \cong K_*$. The spectrum $\mathrm{Hom}(I, I)$ is $K$-local (6.3), is not contractible, and is an $\mathbb{S}$-algebra under composition; it follows that the unit map $\mathbb{S} \to \mathrm{Hom}(I, I)$ is nontrivial on $K_*$. Visibly, then, the unit map is an isomorphism on $K_*$ and induces an equivalence $\hat{\mathcal{S}} \to \mathrm{Hom}(I, I)$. It is not hard to identify this equivalence with the natural map $\hat{\mathcal{S}} \to D_I^2(\hat{\mathcal{S}})$ and conclude that $\kappa_{\hat{\mathcal{S}}}$ is an equivalence. □

*Proof of 6.1.* Suppose that $I$ is invertible. Use 4.8 to deduce

$$\mathrm{Hom}(K, I) \sim \mathrm{Hom}(K, \hat{\mathcal{S}}) \hat{\otimes} I$$

and observe that both $\mathrm{Hom}(K, \hat{\mathcal{S}})$ (1.17) and $K \hat{\otimes} I$ (4.3) are equivalent to $K$ up to suspension. The conclusion is that $\mathrm{Hom}(K, I)$ is equivalent to $K$ up to suspension.

Suppose on the other hand that $\mathrm{Hom}(K, I)$ is equivalent to $K$, up to suspension. It follows from 6.4 that the natural map

$$\mathrm{Hom}(K, \hat{\mathcal{S}}) \sim \mathrm{Hom}(K, D_I^2 \hat{\mathcal{S}}) \to \mathrm{Hom}(D_I \hat{\mathcal{S}}, D_I K) \sim \mathrm{Hom}(I, D_I K)$$

is an equivalence. The conclusion is that $K^*I$ is isomorphic to $K_*$, up to suspension, and hence by 4.3 that $I$ is invertible.

For the rest of this section, $\mathcal{E}$ will denote the endomorphism $\mathbb{S}$-algebra $\mathrm{End}(E)$ of $E$. The left action of $E$ on itself gives a ring map $E \to \mathcal{E}$.

6.5. **Proposition.** *Suppose that $E'$ is any right $\mathcal{E}$-module which is equivalent as an $E$-module to $E$. Then $E'$ is finitely built from $\mathcal{E}$ as a right module over $\mathcal{E}$.*

*Proof.* Consider two right actions $E(1)$ and $E(2)$ of $\mathcal{E}$ on $E$ which extend the right action of $E$ on itself. Since $\mathcal{E}$ is in fact the endomorphism $\mathbb{S}$-algebra of $E = E(1)$, the right action of $\mathcal{E}$ on $E(2)$ is determined by an $\mathbb{S}$-algebra homomorphism $\alpha : \mathcal{E} \to \mathcal{E}$. For any right $\mathcal{E}$-module $M$, let $M^\alpha$ denote the right $\mathcal{E}$-module obtained by twisting the action of $\mathcal{E}$ on $M$ by $\alpha$, so that $E(2) = E(1)^\alpha$. As in [11, §7], the homomorphism $\pi_*(\alpha) : E_*[\![\Gamma]\!] \to E_*[\![\Gamma]\!]$ is determined by a cocycle representing an element of $H^1(\Gamma; E_0^\times)$, and in particular, $\pi_*(\alpha)$ is an isomorphism. It follows that if $M$ is a free right $\mathcal{E}$-module, so is $M^\alpha$; if $M$ is finitely built from $\mathcal{E}$ as a module over $\mathcal{E}$, so is $M^\alpha$. It suffices then to find a single example of a suitable $E(1)$ which *is* finitely built from $\mathcal{E}$. For this, take $E(1) = \Sigma^{n^2}\hat{D}E$; the distinction between the left action of $E$ on $\hat{D}E$ (4.9) and the corresponding right action is immaterial, since $E$ is a commutative $\mathbb{S}$-algebra. Since $\hat{\mathcal{S}}$ is finitely built from $E$ (2.11), $\mathrm{Hom}(E, \hat{\mathcal{S}}) = \hat{D}E$ is finitely built from $\mathrm{Hom}(E, E) = \mathcal{E}$ as a right module over $\mathcal{E}$. $\square$

6.6. **Theorem.** *The functor $I \mapsto \mathrm{Hom}(E, I)$ gives a bijection between equivalence classes of invertible $K$-local spectra and equivalence classes of right $\mathcal{E}$-modules which are equivalent to $E$, up to suspension, as right $E$-modules.*

6.7. *Remark.* The inverse bijection sends a right module $E'$ of the indicated type to $E' \otimes_{\mathcal{E}} E$.

*Proof.* First observe that if $I$ is an invertible $K$-local spectrum, then $\mathrm{Hom}(E, I)$ is equivalent to $E$ as a right $E$-module: this follows from 4.3, together with the fact (4.8, 4.9) that there are equivalences

$$\mathrm{Hom}(E, I) \sim \mathrm{Hom}(E, \hat{\mathcal{S}}) \hat{\otimes} I \sim \Sigma^{-n^2} E \hat{\otimes} I\,.$$

Next, we claim that for *any* $\hat{\mathcal{S}}$-module $X$, in particular for $X = I$, the natural map

$$\mathrm{Hom}(E, X) \otimes_{\mathcal{E}} E \to X$$

is an equivalence. To see this, fix $X$, and consider the class of all spectra $Y$ such that the natural map

$$\operatorname{Hom}(E,X)\otimes_{\mathcal{E}}\operatorname{Hom}(Y,E)\to\operatorname{Hom}(Y,X) \tag{6.8}$$

is an equivalence. This class certainly includes $Y=E$. Since both sides of 6.8 respect cofibration sequences, and $E$ finitely builds $\hat{\mathcal{S}}$ [13, 8.9, p. 48], the class includes $Y=\hat{\mathcal{S}}$, which gives the desired result (cf. [5, 2.10]).

Now suppose that $M$ is a right $\mathcal{E}$-module which is equivalent to $E$ as a right $E$-module. Let $Y=M\otimes_{\mathcal{E}}E$. We will show that $Y$ is invertible, and that the natural map

$$M\sim M\otimes_{\mathcal{E}}\operatorname{Hom}(E,E)\to\operatorname{Hom}(E,M\otimes_{\mathcal{E}}E)=\operatorname{Hom}(E,Y)$$

is an equivalence. For the second statement, consider the class of right $\mathcal{E}$-modules $X$ with the property that the natural map

$$X\sim X\otimes_{\mathcal{E}}\operatorname{Hom}(E,E)\to\operatorname{Hom}(E,X\otimes_{\mathcal{E}}E) \tag{6.9}$$

is an equivalence. The class certainly includes the free module $X=\mathcal{E}$, and hence, by a thick subcategory argument, all modules finitely built from $\mathcal{E}$. By 6.5, $M$ is finitely built from $\mathcal{E}$, and so the class includes $M$. Again because $M$ is finitely built from $\mathcal{E}$, $Y$ is finitely built from $\mathcal{E}\otimes_{\mathcal{E}}E\sim E$, and so (4.12) the natural maps $E\to\hat{D}^2E$ and $Y\to\hat{D}^2Y$ are equivalences. This gives an equivalence

$$M\sim\operatorname{Hom}(E,Y)\sim\operatorname{Hom}(\hat{D}Y,\hat{D}E)\sim\operatorname{Hom}(\hat{D}Y,\Sigma^{-n^2}E)\,,$$

where the last equivalence is from 4.9. By 4.3(5), $\hat{D}Y$ is invertible, and so $Y=\hat{D}(\hat{D}Y)$ is also invertible (4.7). □

*Proof of 1.25.* By 2.8 and 2.12, a spectrum $X$ which is built from $K$ is a Brown-Comenetz dualizing module for $\mathcal{S}\to K$ if and only if $\operatorname{Hom}(K,X)$ is equivalent up to suspension to $K$. It then follows from 2.17 and 6.1 that the assignment $X\to\hat{X}$ gives a bijection between equivalence classes of such Brown-Comenetz dualizing modules and invertible $K$-local spectra; the inverse bijection sends $Y$ to $\operatorname{Cell}_k Y$. The proof is completed by invoking 6.6 □

6.10. *Remark.* One could consider the *moduli space* **Pic** of invertible $K$-local spectra; this is the nerve of the category whose objects are the invertible $K$-local spectra and whose morphisms are the equivalences between them [3]. Up to homotopy **Pic** can be identified as a disjoint union $\coprod_\alpha B\operatorname{Aut}(I_\alpha)$, where $I_\alpha$ runs through the equivalence classes of invertible modules, and $\operatorname{Aut}(I_\alpha)$ is the group-like simplicial monoid of self-equivalences of $I_\alpha$. The space **Pic** is an associative monoid, even

an infinite loop space, under a product induced by $\hat{\otimes}$; its group of components is the Picard group considered in [11]. Let $E^\times$ denote the group of units of the ring spectrum $E$, so that $\pi_0 E^\times \cong E_0^\times$ and $\pi_i E^\times \cong \pi_i E$ for $i > 0$. It seems that one can construct a second quadrant homotopy spectral sequence

$$E^2_{-i,j} = H^i_c(\Gamma, \pi_j BE^\times) \Rightarrow \pi_{j-i}\mathbf{Pic}$$

which above total degree 1 agrees up to a shift with the Adams spectral sequence for $\pi_*\hat{\mathcal{S}}$ (compare the proof of 4.9). This agreement is not surprising, since each component of **Pic** is $B\mathcal{S}^\times$. The edge homomorphism $\pi_0\mathbf{Pic} \to H^1_c(\Gamma, E_0^\times)$ is the map used to detect Picard group elements in [11]. The obstructions mentioned in [13, p. 69] seem to be related to the first $k$-invariant of $B\mathbf{Pic}$ (for associative pairings) or to the first $k$-invariant of the spectrum obtained by delooping **Pic** (for associative and commutative pairings).

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